Two-dimensional acoustic turbulence

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Two-dimensional turbulence of the waves with linear dispersion law is analyzed numerically at small Mach numbers and large Reynolds numbers. It is shown that the energy-flux relation is close to $E\propto P^{2/3}$ as for a one-dimensional system. The analysis of the wave distribution in **k** space shows that the anisotropic large-scale pumping produces turbulence as a set of narrow jets that do not smear as the cascade proceeds towards high wave numbers. The energy spectrum along the direction of a jet is close to $E(\mathbf{k}_\parallel)\propto\mathbf{k}_\parallel^{-2}$ due to shock waves, while the spectrum per unit interval of wave numbers is $E(k)\propto P^{2/3}k^{-1}$ contrary to all previous predictions. Probability density functions of the velocity and velocity differences are found and compared with recent theoretical predictions. [S1063-651X(96)03709-9]

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The nonequilibrium state of a continuous medium under an excitation of long density perturbation is called acoustic turbulence. Long waves on shallow water also have a linear dispersion relation $\omega_k=ck$ and can be considered as two-dimensional sound. The energy cascade is expected towards small scales where dissipation takes place. Developing a description of acoustic turbulence is a long-standing problem in turbulence theory [1–5]. Even for a small Mach number, the absence of the wave velocity dispersion makes nonlinear interaction strong. Waves moving in the same direction have the same velocity and interact strongly; any one-dimensional perturbation thus breaks and creates a shock.

The puzzle now is whether or not the weak interaction with a turbulent background (i.e., with waves moving in different directions) produces some kind of turbulent viscosity that may prevent any given wave from breaking. In other words, are plane shocks stable with respect to finite two-dimensional perturbations? If plane shocks survive at developed two-dimensional turbulence, what is their angular distribution? As is usual in turbulence theory, there are quite a few predictions for the steady turbulence spectrum. Here we list them. Considering waves with small dispersion $\omega_k=ck(1+a^2k^2)$ and even smaller Mach numbers ($M\ll ak\ll 1$) one can apply weak turbulence theory and get the spectrum $E_k\propto P^{1/2}k^{-d/2}a^{(3-d)/2}$, where d is the space dimensionality and flux P is the density of the energy dissipation rate [4]. For $d=3$, the dispersion parameter a disappears which led Zakharov and Sagdeev to suggest the spectrum $E_k\propto P^{1/2}k^{-3/2}$ for the dispersionless limit as well [2]. This is probably the case at least for not very large k where infrared divergences do not yet manifest themselves [6]. In two dimensions, the weakly turbulent solution vanishes in the dispersionless limit. One may thus turn to dimensional analysis. Assuming shocks to be absent, different hypothetical steady spectra have been suggested from either dimensional reasoning ($E_k\propto P^{4/7}k^{-11/7}$ [2]) or analysis of the one-loop approximation ($E_k\propto P^{3/5}k^{-17/12}$ and $E_k\propto P^{3/5}k^{-8/5}$ [7]). For the complete self-similarity being assumed, the turbulence spectrum can be written as follows: $E_k=\rho\omega_k^2k^{d-6}f(Pk^{5-d}/\rho\omega_k^3)$ up to unknown dimensionless function $f(\xi)$ [4]. For example, requiring frequency to disappear from that relation, one gets $f(\xi)\propto\xi^{2/3}$ and Kolmogorov spectrum for incompressible fluid. Requiring $f=$ const, one gets universal spectra (k^{-1} for three-dimensional (3D) acoustics, for instance [5]). For $d=2$, we substitute $\omega_k=ck$ and get $E_k\propto k^{-2}$ as for the spectrum of shock waves. Indeed, if one assumes that acoustic turbulence is a set of isotropically distributed locally plane shocks, then in the spirit of Kadomtsev and Petviashvili [1] one can get $E_k\propto k^{-2}$ for any dimensionality. The flux dependence for the shock spectrum could be found from the following simple estimates. The flux P relates to the energy E by a typical transfer time τ which is of the order of breaking time $P\simeq E/\tau\simeq Eu/L$. Here u is a rms velocity perturbation and L is a typical distance between shocks. In the last formula, only u is related to E. According to Kadomtsev and Petviashvili [1], the total energy is given by the set of jets each with the solid angle $\delta\Omega\simeq(k_\perp/k)^{d-1}\simeq(u/c_s)^{(d-1)/2}$.

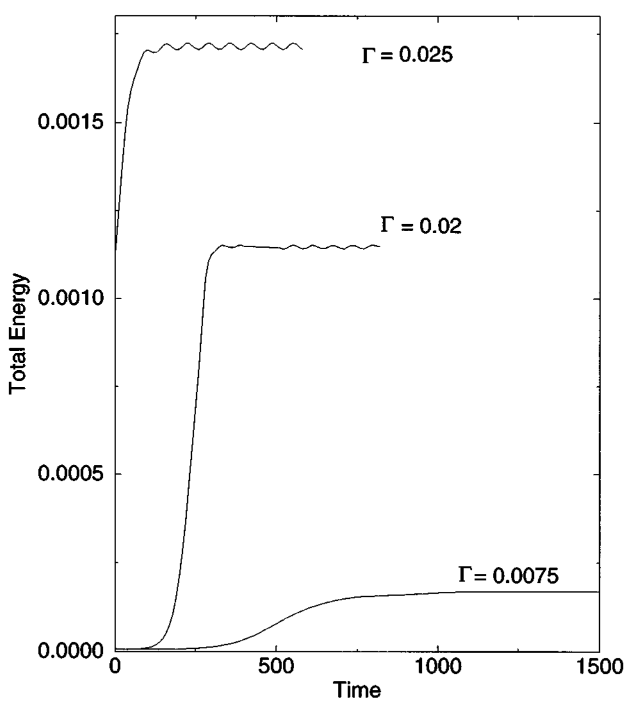

FIG. 1. Evolution of the total energy.

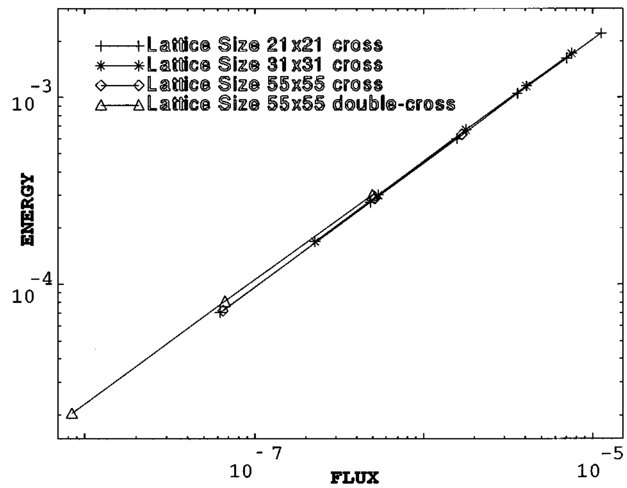

FIG. 2. Energy-flux dependence.

Since $(u/c)^2 \simeq E\,\delta\Omega/\rho c^2$ then

$$E_k \propto P^{(5-d)/(7-d)} k^{-2}. \tag{1}$$

To summarize, if we denote $E_k \propto P^{\alpha} k^{-\beta}$ then there are the following predictions for two-dimensional acoustic turbulence: $(\alpha,\beta)=(\frac{4}{7},\frac{11}{7}),(\frac{3}{5},\frac{17}{12}),(\frac{3}{5},\frac{8}{5}),(\frac{3}{5},2)$. All those spectra are isotropic. However, it is practically impossible to generate ideally isotropic large-scale pumping. As the cascade proceeds towards high k, it may become either more or less isotropic depending on the dispersion sign [4,8]. In the dispersionless limit, the angular shape of the spectrum has to be given by pumping (at least until such a high wave number where nonlinear frequency renormalization provides for some dispersion).

In this paper we numerically analyze two-dimensional acoustic turbulence in a square box. We show below that none of the above predictions is correct for the turbulence produced by the pumping having the symmetry of the box (cross and double cross in **k** space). The spectrum is close to $E_k \propto P^{2/3} k^{-1}$. Therefore, $\alpha=\frac{2}{3}$ which corresponds to (1) with $d=1$ as for Burgers equation [9]. We show that the spectrum is indeed a set of quasi-one-dimensional jets. The number of jets is determined by the pumping symmetry and is not given by $2\pi/\delta\Omega$.

The energy of the compressible fluid becomes Hamiltonian in the variables density ρ and potential $\Phi=(\nabla \mathbf{v})$. For small variations $\rho=\rho_0+\rho_1$, $\rho_1 \ll \rho_0$, the Hamiltonian is

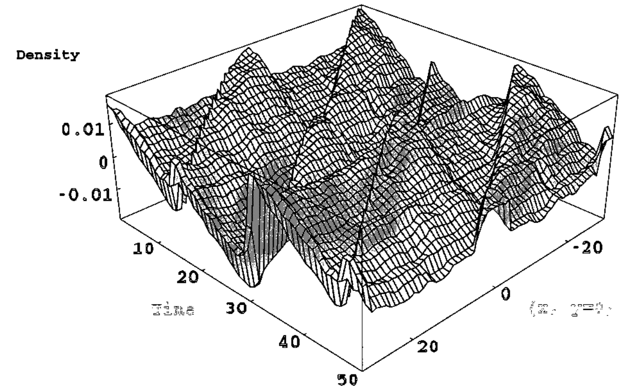

FIG. 4. Density profile for cross pumping.

$H=(\frac{1}{2})\int[\rho|\nabla\Phi|^2+c^2\rho_1^2/\rho_0-c^2\rho_1^3/3\rho_0^2]d\mathbf{x}$ with the sound velocity c [4]. Assuming periodic boundary conditions, we consider turbulence in the square box in **k** space. Choosing units with $c=\rho_0=1$ and introducing the complex amplitude $a(\mathbf{k})=\sqrt{1/2k}[\rho_1(\mathbf{k})+ik\Phi(\mathbf{k})]$, we get the Hamiltonian density

$$\mathcal{H}=\sum_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2}[k|a_k|^2+V_{k12}\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)(a_k^* a_1 a_2+\text{c.c.})]$$

with $V_{k12}=(\cos\theta_{k1}+\cos\theta_{12}+\cos\theta_{k2}-1)(kk_1k_2/4)^{1/2}$ [4]. We solve the dynamical equations

$$\partial a(\mathbf{k},t)/\partial t=-i\,\delta\mathcal{H}/\delta a^*(\mathbf{k},t)+\Gamma(\mathbf{k})a(\mathbf{k},t), \tag{2}$$

$\Gamma(\mathbf{k})$ describes the external influence. Positive $\Gamma(\mathbf{k})$ corresponds to the pumping present at small k: either cross with $\Gamma(\mathbf{k})=\Gamma$ at four points $(k_x,k_y)=(0,1),(1,0),(-1,0),(0,-1)$ or double cross with four diagonal points added that have $\Gamma(1,1)=\Gamma/2$. We took the numerical values $10^4\Gamma=(50,75,100,150,200,250)$ at different runs. We use also two forms of damping with the same amplitude but different widths: narrow [when $\Gamma(\mathbf{k})=-10\Gamma$ only for the points on the boundary and next to them i.e., for either k_x or k_y equal to N and $N-1$] and wide [when $\Gamma(\mathbf{k})=-10\Gamma$ for the one third of the domain i.e., for either k_x or k_y lying in the interval $[2N/3,N]$]. The latter has been done to exclude the effect of aliasing on the spectrum. In the rest of **k** space, $\Gamma(\mathbf{k})=0$ which corresponds to the inertial interval of turbulence. The nonlinear term in the equation was calculated in **r** space by using fast-Fourier transforms.

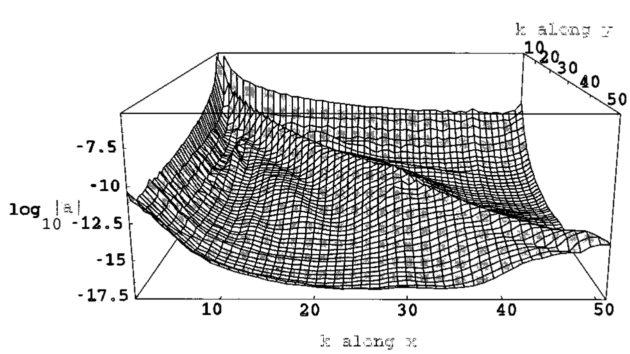

FIG. 3. $|a(\mathbf{k})|$ for double cross pumping.

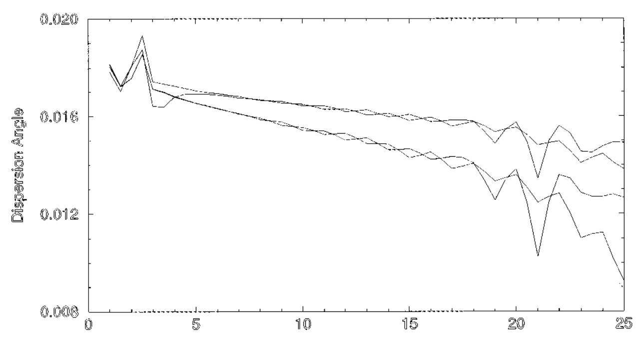

FIG. 5. Dispersion angle $\Delta\theta(k)$.

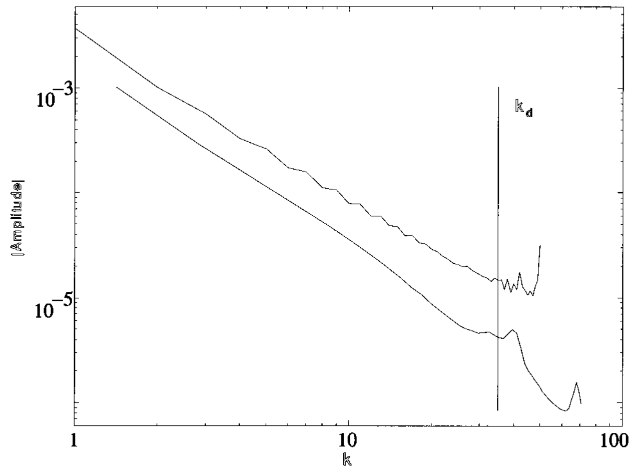

FIG. 6. Amplitude versus k along the jets.

We exploited the symmetry $a(k)=a(-k)$ and made the computations in the quadrangle $0 \leqslant k_x, k_y \leqslant N$ with lattice size $2N+1$ being 21,31,55,101 for different runs. All figures are in dimensionless units.

Figure 1 shows the evolution of the total energy. The oscillations around steady state are due to nonlinear interaction (the frequency behaves approximately as $\nu \propto E \propto \Gamma^2$ for not very large pumping); they are most probably due to periodic energy exchange between two groups of waves, those connected with the pumping and the rest of the spectrum. Such oscillations were first predicted in [10] and were later seen in numerical modeling of optical turbulence [11].

The mean energy in the steady state is plotted against flux on Fig. 2 for different sizes of the lattice and different pumping. All data agree well with the dependence $E \propto P^{2/3}$ typical for a one-dimensional set of shocks.

Indeed, one may see from Fig. 3 (in **k** space) as well as from Fig. 4 (in **r** space) that the turbulence is a set of narrow jets (four for the cross pumping and eight for double cross) of shock waves. Figure 3 is in log-log scale; it is clearly seen that the level of turbulence outside the jets is few orders of magnitudes less. The directions of the jets are determined by pumping at $k=1$ or $k=\sqrt{2}$, the angular width of a jet does not increase (even slightly decreases) as the cascade proceeds towards large k—Fig. 5. Different initial data were chosen. All the results presented are independent of the initial state.

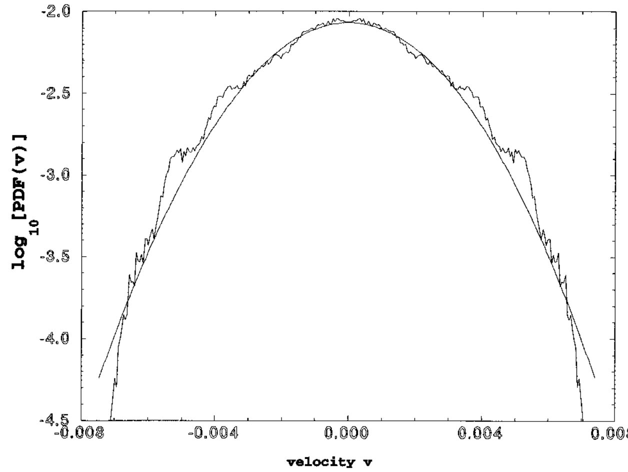

FIG. 8. Logarithm ($\log_{10}$) of velocity PDF for double cross pumping.

The k dependence of the wave amplitudes along the jet axis is plotted on Fig. 6 for the run with the lattice size 101×101 and a double cross pumping. The upper curve corresponds to the main jet along x while the lower one corresponds to the diagonal jet. The energy spectrum $E(k_\parallel)=k_\parallel |a(\mathbf{k})|^2$ is close to $k_\parallel^{-2}$ as due to shocks. Some flattening of the upper curve near the dissipation cutoff is probably due to the bottleneck phenomenon [4,12].

The shocks also manifest themselves at the probability density functions of the velocity and density. One-point pair distribution function (PDF) of the velocity x component is presented on Figs. 7 and 8.

It is seen that the small fluctuations in between the shocks (responsible for the PDF near the top) have Gaussian statistics because the parabolic fit (solid line) works well. Shocks are responsible for two side maxima in PDF and the respective deviation from the Gaussian statistics. For a cross pumping, the shocks are of more or less the same amplitude and move with the same velocity (which also could be seen from

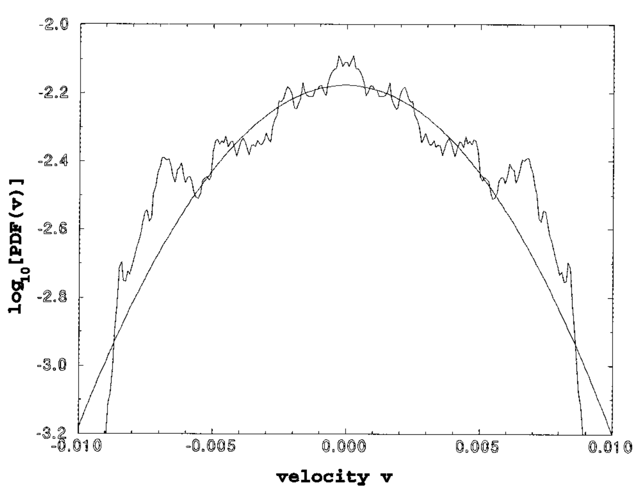

FIG. 7. Logarithm ($\log_{10}$) of velocity PDF for cross pumping.

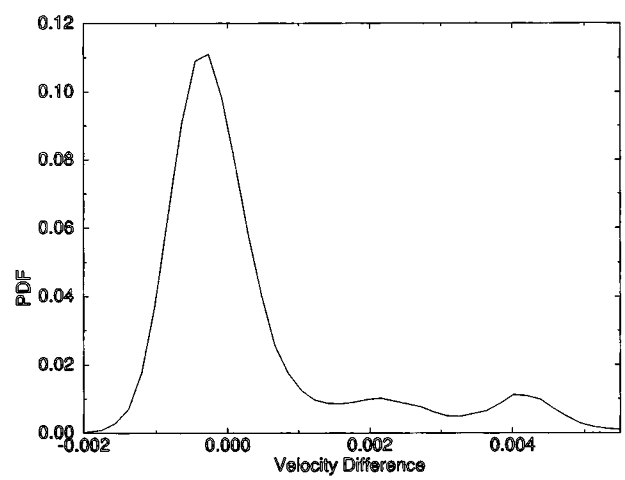

FIG. 9. PDF of velocity differences $\Delta v(r)$ at $r=3$ for double cross pumping.

Fig. 4). The positions of the local side maxima at the velocity $v \approx \pm 0.007$ exactly correspond to the velocity of the fronts seen at Fig. 4. For a double cross pumping where oblique shocks (along the diagonal of the box) are generated as well, there are two side maxima in the one-point PDF on Fig. 8 that correspond to the x components of the velocities of the direct and oblique shocks ($v \approx \pm 0.0035$ and $v \approx \pm 0.0055$).

The histogram of the velocity differences are presented in Fig. 9. It is asymmetric similarly to that which has been observed for one-dimensional turbulence [13]. Again, there are two side maxima for the double cross pumping, the maxima correspond to the velocity jumps in a direct and oblique shock, respectively. There is only one side maximum for the cross pumping (not shown). The left wing decays faster than Gaussian; it could be fitted by $\exp[-|\Delta v|^{\nu}]$ with ν close to 3. Note that the functional dependence $\exp[-|\Delta v|^3]$ has been predicted by Polyakov for 1D Burgers equation within the framework of a self-consistent conjecture on the structure of the point-splitting anomaly [14]. The same asymptotic law for Burgers equation has been obtained by Gurarie and Migdal [15] by the instanton formalism suggested for turbulence in [16], the instanton configuration being a linear velocity profile. Therefore, the left wing of the velocity difference PDF given by the linear pieces of the velocity profile is universal while the right wing depends on pumping that prescribes the distribution of the shock amplitudes.

To conclude, our numerics suggest a simple consistent picture of acoustic turbulence at small Mach and large Reynolds numbers: a general anisotropic pumping produces jets of shock waves moving at the directions of pumping maxima. The angular widths of the jets decreases as k increases. The energy spectrum is due to one-dimensional (plane) fronts $E(\mathbf{k}_\parallel) \propto \mathbf{k}_\parallel^{-2}$ so that the local Mach number v_k/c decreases with k. The decrease of the angular width and Mach number makes it unlikely that the regime will qualitatively change as k increases. We demonstrated that the shocks are stable and they determine turbulence down to the viscous scale. The main result of the paper is the macroscopic relation $E \propto P^{2/3}$, thus shown to be valid for 2D as well as for 1D, it is insensitive to the behavior at high k because the most of the energy is contained at small k.

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